Fractals, Vol. 2, No. 6 (1994)

Complexity Maps Reveal Clusters in Neuronal Arborizations

B. Dubuc, S. W. Zucker
Center for Intelligent Machines,
McGill University,
3480 University St.,
Montréal, (Québec) Canada H3A 2A7

M. P. Stryker
Department of Physiology and Neuroscience,
University of California,
San Francisco, CA 94143-0444 USA

A central issue in characterizing neuronal growth patterns is whether their arbors form clusters. Formal definitions of clusters have been elusive, although intuitively they appear to be related to the complexity of branching. Standard notions of complexity have been developed for point sets, but neurons are specialized "curve-like" objects. Thus we consider to problem of characterizing the local complexity of a 'curve-like' measurable set. We propose an index of complexity suitable for defining clusters in such objects, together with an algorithm that produces a complexity map which gives, at each point on the set, precisely this index of complexity. Our index is closely related to the classical notions of fractal dimension, since it consists in determining the rate of growth of the area of a dilated set at a given scale, but it differs in two significant ways. First, the dilation is done normal to the local structure of the set, instead of being done isotropically. Second, the rate of growth of the area of this new set, which we named 'normal complexity', is taken at a fixed (given) scale instead instead of around zero. The results will be key in choosing the appropriate representation when integrating local information in low level computer vision and, more specifically, they lead to the quantification of axonal and dendritic tree growth.

1. INTRODUCTION

Measures for curves, dimensionality, and complexity are coupled notions, and the relationships between them are important practically as well as theoretically. Measures of curves might include their length, the number of components, or the area covered. However, the situation is more subtle than this, as is illustrated in Fig. 1. The first example is a demonstration due to Kanisza,[1] in which a pin-striped surface appears to be occluding a rectangle. Thus curves, or sets of curves, can actually connote either the outline of objects (as in the rectangle) or surfaces (the pin stripes). Closer examination reveals that the rectangle is actually continuous through the surface, however, suggesting that the visual inferences somehow group the pinstripes together and ignore the fact that the rectangle is the longest curve in the image. This example clearly shows that neither the length of the curve (defining the rectangle) nor the number of components (defining the pinstripes) is dominant. If another pinstripe were added, the percept would not change. Moreover, the saliency of the rectangle is camouflaged. The second example is the axonal tree of a neuron in developing visual cortex.[2] Clustered interconnections between neurons are important physiologically, but again neither length, branching factors, nor self-intersections suffice to define it.

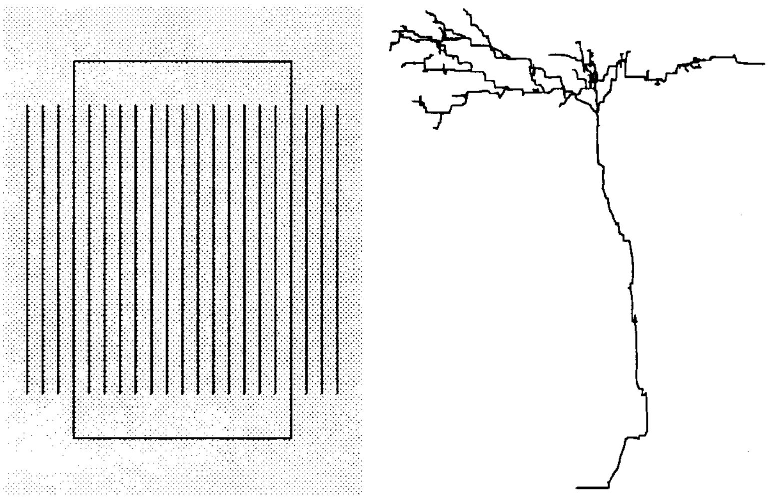

Fig. 1 Two examples of curve-like sets. Left: Kanisza pattern; right: reconstruction of a neuron. Both of these illustrate the need of using different representations within the same curve-like set. Notice how difficult it is to perceive the rectangle in the Kanisza pattern and how tedious it is to count the number of lines in the dendritic cluster of the neuron.

The goal of this paper is to provide a local measure of the complexity both in space and in scale for curve-like sets (sets with finite Hausdorff 1-measure).[3] It will provide, among others, a tool to properly separate regions in curve-like patterns that fill areas from those that extend mainly along their length (also called here curvi-representable).

2. MAPPING THE (NORMAL) COMPLEXITY OF CURVE-LIKE SETS

2.1 Local structure

The development of our new complexity measure is based on the use of the local structure of a set E and on the assumption that the set is rectifiable. First, we need to define a mathematical structure that gives the 'orientation' of the set E at each point:

Definition 1 (Tangent map) *Given a curve-like set E, we define the* tangent map τ *to be*

$$\tau = \bigcup_{x \in E} (x, \Theta(x)), \tag{1}$$

where $\Theta(x)$ is the set of tangents (in the Besicovitch sense) at x.

This definition allows the existence multiple tangents at a point, a key requirement in the type of sets we are studying when doing curve detection. The set E being rectifiable, we know that these do not occur too often since there is a unique tangent almost everywhere.

2.2 Oriented dilations

Minkowski dilations are routinely used in mathematical morphology[4,5] and for the estimation of fractal dimension.[3,6] The approach consists in creating a new set which is the Minkowski sum with a dilating (structuring) element, the result being commonly called, the *Minkowski sausage.*[6]

One of the key differences between our approach and the standard fractal analysis techniques is the fact that the dilations will not be done isotropically but will adapt to the local structure of the set. We call them *oriented dilations* and we can show that they are necessary for separating sets of different complexity.[7]

Definition 2 (Normal dilation) *Let E be a curve-like set and τ its tangent map. The* normal dilation $E_N(\epsilon)$ *of E at a scale ϵ is the dilation of the set E with the segment $(-\epsilon, \epsilon)$ in the direction normal to the tangents $\theta \in \Theta(x)$ at x.*

This departure from the standard Minkowski dilation approach will be essential for our analysis since it will segregate the classification 1D-2D (using normal dilation) from the one of 0D-1D (using tangential dilations).

2.3 Normal complexity, index, and map

The local information contained in the tangent map τ can be used to calculate the *normal complexity* $C_N(\epsilon)$ of a studied object at a given scale ϵ. The main idea is to look at the rate of growth of $|E_N(\epsilon)|_2$ at scale ϵ. If the area of $E_N(\epsilon)$, denoted $|\cdot|_2$, is of order α, we say that the *normal complexity* $C_N(\epsilon)$ for E is $2-\alpha$.

Definition 3 (Normal complexity) *Let E be a curve-like set. If $E_N(\epsilon)$ is the normal dilation of E at scale ϵ, and if $|\cdot|_2$ denotes its area, then the* normal complexity $C_N(\epsilon)$ *at scale ϵ will be the left derivative of the graph* $\left(\log\left(\frac{1}{\epsilon}\right), \log\left(\frac{1}{\epsilon^2}|E_N(\epsilon)|_2\right)\right)$.

From the structure of curve-like sets, one can show that the normal complexity is indeed well-defined.[7] For this we first show that $|E(\epsilon)|_2$ is a concave function of ϵ and then, using a standard result from analysis, we get that the left and right derivatives exist everywhere, therefore justifying the existence of the normal complexity at ϵ.

So far the normal complexity was computed globally for the entire set E. We would rather like to compute it locally, i.e. calculate the complexity at a given point within a compact region $\Omega(x)$. Computing the area of the (normal) dilated set restricted to the compact region $\Omega(x)$, and estimating the rate of growth from log-log data gives us a number that we call the *normal complexity index.* The bundle of all these provides the *normal complexity map.*

3. APPLYING TO CURVE-LIKE SETS

We have computed the complexity map for the the Kanisza pattern. Results are displayed in Fig. 2 (left) where the normal complexity index is mapped as a grey value. Dark values stand for low complexity while light refer to points within a complex region. From this we see that, at the studied scale, the patch stands out as requiring a different representation from the curvi-representable part (the "curves" in the top and in the bottom).

The technique was also applied to the axonal reconstruction data. The normal complexity map is shown in Fig.2 (right). Once more, it is obvious that the normal complexity succeeds in discriminating places where the pattern extends along its length from places where it tends to form a cluster. The cluster clearly stands out in white suggesting that at the scale studied a different representation should be chosen to represent it than the axon.

4. CONCLUSION

The transition from local representations to global ones is a key problem in computer vision. In this paper, we have presented an intermediate representation scheme for the local structure of a curve-like set – the tangent map – and have demonstrated that the local complexity was key to determine the representation underlying the integration process.

The example with the Kanizsa pattern was to that extent very informative. Within one scale, one observes two different types of objects coexisting within the scene: a curvi-representable object (the top and bottom part of a hollow rectangle) and a grating patch. The normal complexity map clearly supported what was observed. In the case of the axonal arbour segmentation, it successfully discriminated the parts that extended along their length from clusters, such as those formed during development.

Our approach differs from the classical ones in fractal analysis[6] in the sense that it studies sets dilated with respect to their local structure. Another key difference is the use of the rate of growth at a particular scale rather than around zero.

The complexity map we built constitutes the first building block of an emerging representational complexity theory for curve-like sets. In the vein of Turing computable numbers,[8] it will now be possible to define the notion of representable curve. Representational complexity theory will generalize the notions presented here and will dictate the action that could be taken before integrating information. It will predict what should be the salient features in the scene and justify unambiguously the choices of possible intermediate representations.

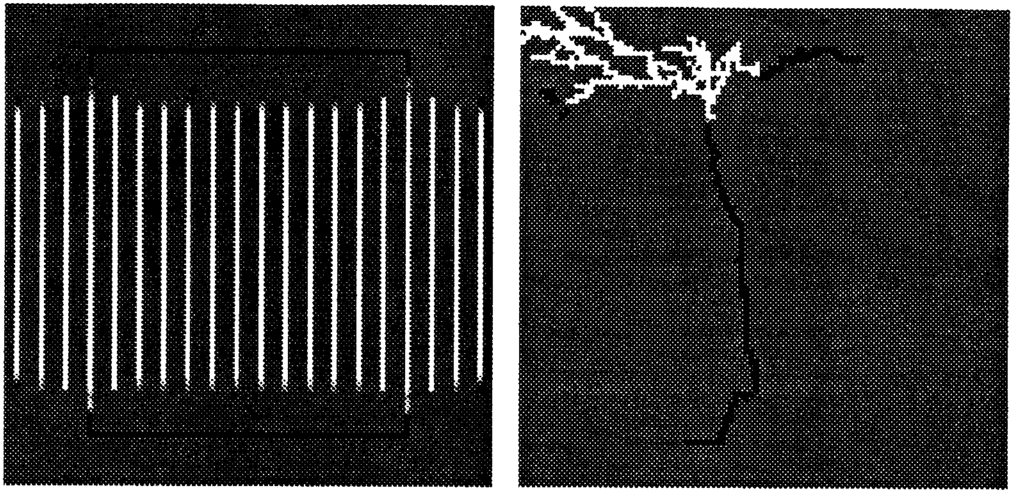

Fig. 2 Computing the normal complexity on discrete tangent maps: for the Kanisza pattern on the left, and for the brain cell on the right. For these two examples, the clustered regions clearly stand out in white.

ACKNOWLEGMENTS

Research supported by the AFOSR and NSERC. S.W.Z. is a Fellow of the Canadian Institute for Advanced Research.

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